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RESEARCH ARTICLE

An Early Algebra Teaching Experiment Aiming at the Improvement of Fourth-Grade Students' Generalization and Justification Strategies

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Abstract

This study aimed to improve fourth-grade students' generalization and justification strategies, and an early algebra teaching experiment was applied for this purpose. In the study, data were obtained through focus group interviews, student worksheets, and video and audio recordings during the research. The data obtained were analyzed by descriptive analysis. In the study, students carried out activities of continuing the pattern, expanding it, completing the missing element, and identifying the pattern rule. It is observed that students used written, verbal, mathematical, and visual representations in their pattern activities. As a result of the study, it was found that students reached generalizations using counting, modeling, iterative, whole-object, contextual, and linear strategies in the teaching experiment. Moreover, it is seen that students used explanation, figural, numerical, and algebraic strategies to verify their generalizations. In line with these results, it is concluded that the early algebra teaching experiment performed with primary school fourth-grade students is effective in students' generalization and justification strategies.

Keywords: Early algebra, generalization strategy, justification strategy, teaching experiment

Introduction

Algebra is regarded as a tool to discover the forms of representation used in areas such as solving equations, arithmetic, and computation and is accepted as a step in discovering appropriate representations for more complex problems (Subramaniam & Banerjee, 2011). Algebra is a system of symbolization and rules used in the problem-solving process at all levels of education, from preschool to high school (Carraher & Schliemann, 2015). In this regard, understanding patterns, their functions, and relationships is one of the purposes of algebra, which takes an important place in school mathematics (National Council of Teachers of Mathematics [NCTM], 2000). It is essential to include pattern activities from the first year of primary school to form a basis for algebra (Herbert & Brown, 1997) because patterns serve as a basis for learning to interpret symbols and help to form and recognize general expressions related to numbers and shapes encountered in algebra at advanced grade levels (Threlfall, 1999). It is considered that the fact that students in the younger age group get acquainted with the pattern and algebra subject during this period and the necessary importance attached to these subjects will ensure that they start their next grade levels with a more solid foundation. Despite the idea that algebra should be included in the primary school curriculum, there are still few studies on adapting algebra to primary school mathematics (Carraher & Schlieman, 2007). This study aimed to contribute to the literature with the teaching experiment to be carried out in primary school mathematics.

When algebra studies conducted with primary school students are reviewed, it is observed that there are different studies on algebra. It is seen that studies have been carried out on early algebra teaching in the pre-algebra period (Fonger et al., 2018; Palabıyık & Akkuş İspir, 2011), discovering and developing functional relationships in students (Cañadas et al., 2016; Ramírez et al., 2020; Xolocotzin & Rojano, 2015), determining the generalization levels of functional relationships (Ayala-Altamirano & Molina, 2021; Kabael & Tanışlı, 2010; Türkmen & Tanışlı, 2019), the effect of pre-algebra teaching activities on academic achievement (Doğan Temur & Turgut, 2020; Turgut & Doğan Temur, 2020), and skills in generalizing and representing functional relationships (Blanton et al., 2015, 2017, 2019; Pinto et al., 2021; Stephens et al., 2017; Ureña et al., 2019). Although studies have been conducted on the pre-algebraic period in the international literature, it is observed that studies on this subject in the national literature are quite limited. In this study, which will be performed for early algebra teaching to be efficient, it is considered that determining fourth-grade students' generalization and justification strategies in pattern tasks in early algebra teaching will make a contribution. To this end, in the present study, a teaching experiment was conducted to improve fourth-grade students' generalization and justification strategies, and the effective generalization and justification strategies of students in this teaching experiment process were examined.

Theoretical Framework

Early Algebra

Algebra is defined as the "essence" of mathematics (Cai & Knuth, 2011). Algebra is regarded as a tool to discover the forms of representation used in solving equations and arithmetic computation and a step

The study was produced from the master's thesis titled "A Teaching Experiment to Develop Algebraic Thinking Structures of 4th Grade Students," which was completed by the first author under the supervision of the second author.

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of discovering representations appropriate for more complex problems (Boyer, 1991). According to Radford (2006), algebra is not only symbolization but also a way of analytical thinking, representing the uncertainty of objects and uncertainty. Kaput et al. (2008) accepted that algebra was related to symbolization and mathematical representation and indicated its relationship with generalization and defined the relationship in the following way, "If there is systematic reasoning in generalizations created using algebraic symbols, this representation is algebraic" (p. 2). Likewise, the association of algebra with generalization was also made by Gattegno (1973), who defined algebra as the symbolization of the generalization reached as a result of arithmetic operations (distribution, combination) and an indicator of the mind's dynamics in this generalization process. Additionally, Carpenter et al. (2003) stated that it would be incomplete to regard algebra as a solution system realized only with algebraic expressions and it also included proof and justification. Kaput (2008) defined the generalization, justification, and symbolization aspects of algebra in a holistic way: (1) generalizing and symbolizing systematically using the traditional symbol system, (2) representing the organized symbolic systems syntactically. This definition of algebra gave rise to the concept of early algebra.

Early algebra is included in primary school, and an opportunity is provided for the development of algebraic thinking by integrating arithmetic and algebra with algebra teaching (Linchevski, 1995). It is stressed that algebra should be included in all levels of education, starting from early childhood (NCTM, 2000). With early algebra, it is ensured that algebra is not only used for symbolization in secondary or high school but is also addressed as a process in which algebraic ideas, concepts, and thinking take place throughout the teaching period (NCTM, 2000). There has been a trend toward early algebra research and studies in recent years (Blanton et al., 2019; Cai & Knuth, 2011; Chimoni et al., 2018; Kaput et al., 2008; Kieran, 2011; Rivera, 2013; Schifter, 2018; Stephens et al., 2017; Pinto & Cañadas, 2021; Vergel, 2015). Whether students are ready to learn algebra at an early age and whether starting to learn algebraic concepts at an early age can solve the difficulties experienced at advanced grade levels are among the reasons for research on early algebra. Mason et al. (2009) revealed the generalization meaning of early algebra and stated that it would not be early to include this understanding in primary school mathematics. Since algebra is effective in many fields of mathematics and at every education level, the definition of algebra has been differentiated, and the idea that algebraic thinking should be extended for a long period from the basic level such as preschool and primary school has emerged (LaCampagne et al., 1995; NCTM, 2000). With this differentiated understanding of algebra, the idea of "early algebra" has been born, and algebra teaching in primary school has emerged.

Early algebra is a way to make important ideas of algebra accessible and interesting to students (Blanton, 2008). Early algebra teaching helps primary school students to form meaning with mathematical structures and reach generalizations by discovering the relationships between these structures themselves (Blanton, 2008). Moreover, with early algebra, it is ensured that students develop an understanding of the concept of variable and represent it in verbal, written, and visual ways, thereby developing an understanding of algebraic concepts (Kaput & Blanton, 2001). Early algebra takes an important place in students' discovery of mathematical structures and relationships (Blanton et al., 2011). Generalization, representation, justification, and reasoning are at the core of early algebra (Blanton et al., 2011; Kaput, 2008).

Generalization

Generalization is one of the most important skills in school mathematics and algebra (Kaput, 2008; Mason, 2018). Generalization is the ability to see the particular in the general (Mason & Pimm, 1984). Accordingly, students should be supported in developing their awareness of seeing the general and discovering generalizations in the context of the specific situations in which they work (Mason, 1996). Emphasizing the relationship between algebra and generalization, Kaput et al. (2008) stated that generalizations based on symbolization, reached as a result of systematic reasoning, had algebraic meaning. With a similar approach, Mason et al. (2009) explained generalization as the lifeblood of mathematics and algebra as the language in which generalizations were expressed. According to these definitions, generalization was a mental process transforming many special cases into a holistic form (Kaput et al., 2008). For example, after a series of examples in which odd and even numbers are summed up, a holistic form is reached when students generalize to the conclusion that the result obtained from the sum of odd and even numbers is always an odd number.

Generalization is the process of defining mathematical structures and relationships (Mason et al., 2009). Stacey (1989) defined the concepts of near generalization (finding the nearest term in the continuation of the pattern) and far generalization (finding a general rule in a pattern) in growing patterns with linear functions and introduced three basic generalization strategies. It is an *iterative strategy* in which the next term of the pattern is obtained from the previous term, it is a *linear strategy* in which a functional relationship is sought, and it is an *expansion strategy* in which proportional reasoning is applied. Radford (2003) classified students' generalization activities as *factual, contextual, and symbolic* generalizations based on pattern studies. Factual generalization is the operational scheme that students use to reach certain terms, although they cannot reach a generalization that applies to all terms. Contextual generalizations are generalizations in which students can refer to the "next term" or a general term, although it is not a valid generalization for all terms. Symbolic generalizations are students' expressing algebraic concepts abstractly without referring to calculation methods or any special terms. Radford (2006) interpreted this classification in an algebraic context. Accordingly, generalizing a pattern algebraically is based on the ability to identify the common feature between terms, realize that the common feature is valid for all terms, and express this situation for any term (Radford, 2006). Lannin (2005) defined generalizations as explicit and non-explicit. Non-explicit strategies are counting and iterative strategies, whereas explicit strategies include whole-object, guess and check, and contextual strategies.

Harel (2001) argues that there are two types of generalization: *generalization of results* based on a few examples and *process generalization* by justifying that it is valid in all terms after reaching a generalization from a few examples. The generalization distinction made by Radford (2003, 2006) adopted the inductive approach, while the generalization classification performed by Lannin (2005) adopted empirical verification. Radford (2006) suggested recognizing the special and using this understanding to integrate it into the general. Additionally, Lannin (2005) divided the types of generalization into visual and numerical representations and indicated that visual representations were a more appropriate type of representation for process generalization.

Generalization is at the center of mathematical activities (Kaput, 1999; Lannin, 2005). The generalization process is a process created on the interaction of individual and collective reasoning with activities supporting generalization (Ellis, 2007, 2011; Jurow, 2004). In this process, generalizations emerge and are expressed verbally and in writing (Ellis, 2007). Generalization is at the center of algebraic thinking as well as mathematical activities (Cooper & Warren, 2011). By including generalization studies in algebra activities, help is provided to students in developing their algebraic thinking, and the process of symbolizing and justifying mathematical expressions is also supported (Blanton & Kaput, 2008, 2011). Thus, it is aimed to improve generalization skills by including students in algebraic activities. Otherwise, if students are not supported in reaching generalizations at an early age, their learning of algebra is also adversely affected (Dienes, 1961). Students' generalization potentials are effective in many algebra activities.

Students have the potential to discover regularities and structures in number and shape patterns, which provides a basis for reaching generalizations and verifying and justifying them (Schifter et al., 2008). With early algebra teaching, students' justification skills, including their generalization skills, are also supported with activities in this context (Breiteig & Grevholm, 2006; Knuth et al., 2002). To this end, in algebra activities supporting generalization, practices consistent with each other and written/verbal interaction environments, teacher–student-problem-representation elements should be connected and interactive (Ellis, 2011).

Justification

Justification ensures that generalizations are addressed as assumptions and, ultimately, these assumptions are verified or falsified (Mason, 2008). Balacheff (2001) defined justification as explanations aiming to make an expression valid for other individuals. Harel and Sowder (1988) introduced three general schemes for justifying mathematical propositions: *appeal to authority*, *verification by examples*, and *generalized arguments*. In an appeal to authority, justification is made based on external sources (textbook, teacher, etc.). In verification by examples, justification is made based on experimental examples. This justification approach may be misleading in some cases, as not taking into account every example. In generalizable arguments, generalizations are reached through justifications based on mathematical expressions, definitions, axioms, and postulates. Balacheff (2001) distinguished justifications as *pragmatic* justifications based on examples and representations and *conceptual* justifications based on abstract formulae or mathematical relationships. Kirwan (2015) defined justifications in his generalizations in the patterns in two ways as verification and explanation.

Providing opportunities for justification in pattern studies to be performed with students will enable the discovery of critical aspects of algebra (Carpenter et al., 2003). When the justification ways of primary school students are reviewed, it is observed that they are generally at the basic level. In studies on students' ways of justification (Carpenter et al., 2003; Schifter et al., 2008), it is seen that they verify with not very complex general forms. For example, students prefer to use drawings or models to justify their generalizations about the pattern structures they have discovered (Schifter et al., 2008).

Method

Qualitative research is a method that provides a holistic examination of phenomena and events in their naturalistic environments (Patton, 2002). In this study, the teaching experiment model, a qualitative research method, was employed since it was aimed to examine the improvement of generalization and justification strategies of fourth-grade students in the process of the algebra teaching experiment. The teaching experiment model has been frequently used in mathematics education studies in recent years. The teaching experiment model was developed by getting inspiration from clinical cases due to the inadequacy of experimental research with the changing understanding of mathematics over time (Steffe, 1983).

In the teaching experiment, answers are sought to the questions of how students make sense of teaching activities and how their knowledge, skills, and experiences change in the teaching process (Sinclair, 1987). The teaching experiment is a comprehensive form of clinical interviews, and it is attempted to make sense of changes in students' knowledge, experience, and mathematical reasoning throughout the process. There are many teaching activities in line with the purpose specified in the teaching experiment (Cobb & Steffe, 2001). Teaching experiments are interventions that are conducted with at least one participant and remind the classroom environment with an increase in the number of participants (Engelhardt et al., 2004). To this end, individual or small group research is carried out in the teaching experiment. The researcher plays the role of teacher and observer in the teaching experiment. Since the teaching experiment is a long-term process, the interaction between the researcher/teacher and participants is quite high (Cobb & Steffe, 2011). In the teaching experiment, the researcher in this role becomes closely acquainted with participants as a result of long-term studies, thus finding the opportunity to look at events or situations from the participants' eyes. The ultimate aim of the teaching experiment is to reveal the development of the existing knowledge and skills with the teaching method applied to participants, design the teaching decisions accordingly, and provide an effective teaching process. The most significant contribution of the teaching experiment to the field of mathematics is that it provides detailed information about the teaching process and guides the research to be done on mathematics teaching (Cobb, 2000). In the current study, it was aimed to improve fourth-grade students' generalization and justification strategies, and a teaching experiment was conducted for this purpose. The workflow of the teaching experiment process performed in the study is shown in Figure 1.

The teaching experiment process consists of sessions covering numerous algebra activities. During these sessions, student worksheets, audio and video recordings, and interview transcripts were

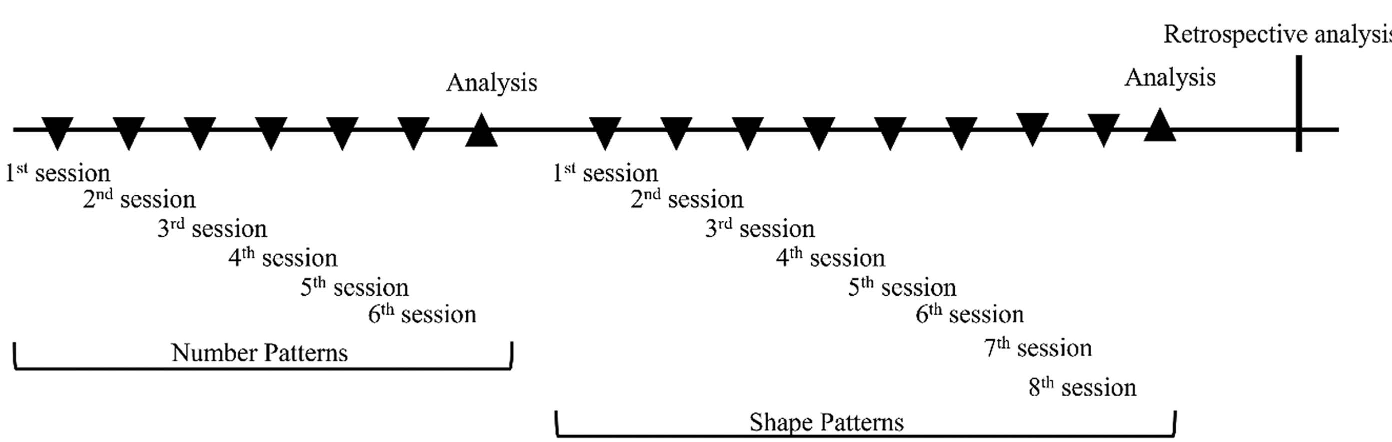


Figure 1.
The Early Algebra Teaching Experiment Process Carried Out in the Study.

used as data collection tools. Throughout the teaching experiment, the researcher interacted with the participants for a long period of approximately 4 months, both as a teacher and as an observer. The researcher/teacher observed changes in the participants at the end of each session and evaluated the working and non-working aspects of the sessions and made improvements. To evaluate the participants' generalization and justification strategies at the end of the teaching experiment, the data obtained using the data collection tools were analyzed with a holistic retrospective analysis method, and the effectiveness of the applied teaching experiment was assessed in the context of the students' generalization and justification strategies.

Study Group

The main purpose of qualitative research is to create a study group from which the richest data can be obtained concerning the research question (Fraenkel et al., 2006). To this end, the participants were included in the study by means of purposeful sampling. Purposeful sampling is the type of sampling that allows a detailed examination of situations varying in terms of knowledge in accordance with the research purpose (Patton, 2002).

The study was conducted with fourth-grade students studying at a public primary school in Erzurum in the fall semester of the 2021–2022 academic year. Six students were included in the study. Criterion sampling, one of the purposeful sampling types, was employed to include students in the study. Criterion sampling is including a person, event, situation, or object with the characteristics determined in line with the research purpose in the sample (Fraenkel et al., 2006). Accordingly, the fact that the participants were fourth-grade students was determined as the inclusion criterion. Since the teaching experiment, whose effectiveness was investigated in the study, was an extracurricular intervention, the research was conducted outside the students' classroom hours. Therefore, voluntary participation was taken as a basis to ensure continuity in the research and the active participation of students in the study.

The current study was conducted with the permission of Yozgat Bozok University Ethics Commission, dated September 29, 2021, and numbered E-39243114-770-37222. Code names were used instead of real names to ensure the participants' confidentiality and obtain the necessary legal permissions in the study's implementation. The individual characteristics of the study participants are presented in Table 1.

When the participants' characteristics are examined, it is seen that they are aged between 9 and 11 and their mathematics achievement is moderate–good. Furthermore, it is observed that there is homogeneity within the group between gender distributions.

Early Algebra Teaching Experiment

In the study, a teaching experiment with growing shape and number patterns was planned and implemented, and the effectiveness of fourth-grade students in generalization and justification strategies was revealed. The scope of the teaching experiment conducted to this end is shown in Table 2.

A teaching experiment, which lasted for approximately 4 months, was conducted within the scope of the study. The teaching experiment activities were prepared by the researchers and finalized in line with the opinions of two researchers, experts in mathematics education. The teaching experiment was prepared and implemented in the context of growing number and shape patterns. Since it is aimed to improve students' generalization and justification strategies in growing pattern studies, the teaching content supporting this improvement was prepared. To this end, activities of creating a pattern, continuing it, completing the missing element, and identifying the pattern rule were carried out. In the process of pattern activities, the use of multiple representations was aimed, and supportive pattern activities were included. Studies were performed to ensure that students discovered the relational situation regarding the pattern structure in number and shape patterns and reached generalizations related to this. In the generalizations reached in pattern activities, it was ensured that students made justifications by aiming to verify and falsify these generalizations. The early algebra teaching experiment was carried out in 15 sessions, each session lasting 40 minutes. The participants carried out the activities in the teaching experiment individually. In each activity, question-answer activities were carried out after the students' individual studies. Thus, it was aimed to reveal the students' views and demonstrate the effectiveness of the targeted skills.

Table 1.
Participants' Individual Characteristics

Participant	Gender	Age	Mathematics Achievement Level
Hale	Female	9	Moderate
Bade	Female	10	Good
Yağız	Male	10	Good
Mert	Male	10	Moderate
Mete	Male	10	Good
Sarp	Male	11	Good

Table 2.
Content of the Early Algebra Teaching Experiment

Pattern Type	Duration	Purpose
Number patterns	6 sessions (40 min × 6 sessions)	• Creates number patterns with a constant difference between them. • Identifies and completes the missing element in number patterns with a constant difference between them. • Expands increasing and decreasing number patterns. • Determines the relationship between numbers with a constant difference between them. • Uses multiple representations in number pattern activities. • Makes a generalization with regard to the rule of growing number patterns. • Makes justifications for generalizations about the pattern rule.
Shape patterns	8 sessions (40 min × 8 sessions)	• Continues geometric shape patterns. • Reaches a generalization for the rule by associating the number of shapes and objects in geometric shape patterns. • Uses multiple representations. • Makes justifications for generalizations about the pattern rule.

Data Collection Tools

In qualitative research, collecting data from many sources provide more detailed, clearer, and more accurate information about the research situation. Many data collection tools were used to examine the investigated problem situation in detail in this study (Yin, 2003). To this end, focus group interviews, student worksheets, and audio and video recordings were included as data collection tools in the teaching experiment process carried out in the study.

Focus Group Interview

The interview is the most basic data collection tool preferred to reach data in qualitative research (Fraenkel et al., 2006). In the present study, focus group interviews were used in the teaching experiment process in which students carried out individual studies. The focus group interview is a data collection tool which is performed between the researcher and the participants in a structured or unstructured way and reveals views for a certain purpose. In this study, since it was aimed to examine the students' generalizations and justifications in the context of teaching elements, it was aimed to share their views and thoughts and reveal the generalization and justification strategies in this process. Six students were included in the focus group interview in the study. The teaching experiment consisted of 15 sessions, and focus group interviews were held in each session.

In the study, after the individual practices were carried out by the students with worksheets, the interview questions were directed to reveal the group view. Care was taken to ensure that the focus group interview questions were easy to understand, simple, and clear questions, to detail the subject with open-ended questions, not to direct students, to ask questions one by one without combining them, to include various questions, and to make a preparation for integrity in the observation form (Fraenkel et al., 2006). The interview form, which was prepared to reveal the students' generalization and justification strategies, was prepared by considering the question of diversity. For this purpose, questions were prepared to reveal the students' generalization, justification, and representation skills in number and shape patterns specific to the content of each session. The prepared interview form was rearranged and finalized in line with the written feedback of two researchers, experts in the field of mathematics (Appendix 1). For the process to be more effective and productive when conducting the interviews, care was taken to ask questions in a conversational style, give feedback on students' answers, and control the process effectively (Patton, 2002). The focus group interviews were recorded using audio and video recording devices. To this end, two video cameras and two voice recorders were kept in the environment where the teaching experiment was conducted. Thus, data loss was prevented.

Documents

Using documents together with interview and observation data tools in qualitative research helps to obtain richer data concerning the research problem (Fraenkel et al., 2006). In this study, various visual-audio elements were included to support the data, test their accuracy, and prevent data loss. The whole teaching experiment process was audio- and video-recorded in the study. For this purpose, a voice recorder and a video camera were kept in the teaching setting. Video recordings allow for the comprehensive analysis of data to better express and justify events or situations. In the current study, video recordings were used to record lectures and teaching activities-based worksheets at the implementation stage. Moreover, the teaching setting and student work carried out in the study were recorded with photographs. Finally, all audiovisual recordings were turned into written documents and used as data collection tools.

Data Analysis

In the teaching experiment carried out to improve the generalization and justification strategies of primary school fourth-grade students, data were obtained using interviews and documents concerning students' generalization and justification strategies. The audio and video recordings obtained during the teaching experiment were transcribed and made ready for analysis. Before starting the analysis, all data documents were filed specifically for each student in each session within the scope of the teaching experiment. Thus, the collected data were made ready for analysis.

The data acquired in the study were analyzed within the scope of the students' resulting and effective generalization and justification strategies. A generalization and justification analysis scheme was used to this end. Analyzing the data obtained in line with the previously determined themes is defined as descriptive analysis (Merriam, 2009). For this purpose, a multistage process was followed in the descriptive analysis conducted in the study (Figure 2).

In the study, first, a search was done on early algebraic thinking in the literature, and generalization/justification components were reached. Afterward, the data set related to the students' generalization and justification strategies in the early algebra teaching experiment conducted in the current study was analyzed. Then all data were read and transferred to the MAXQDA program to perform the analysis. Afterward, the data were coded within the scope of the analysis framework determined based on the literature. The analysis schemes and sample coding used in the study are given in Table 3.

The themes were reached as a result of the coding. Finally, the data were visualized using the MAXQDA program, interpreted with direct quotations, and reported in the findings section.

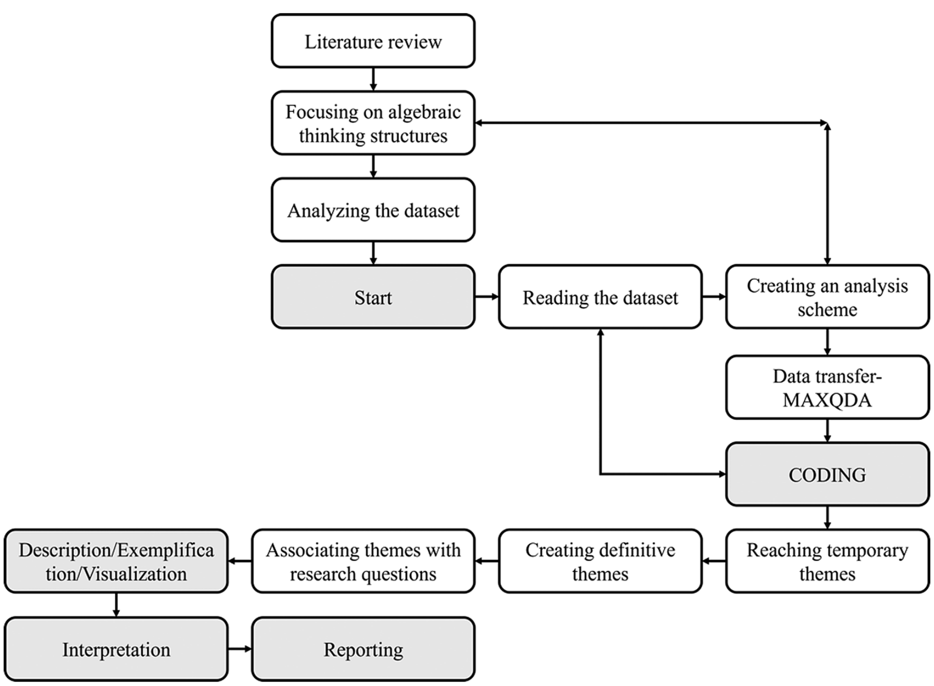
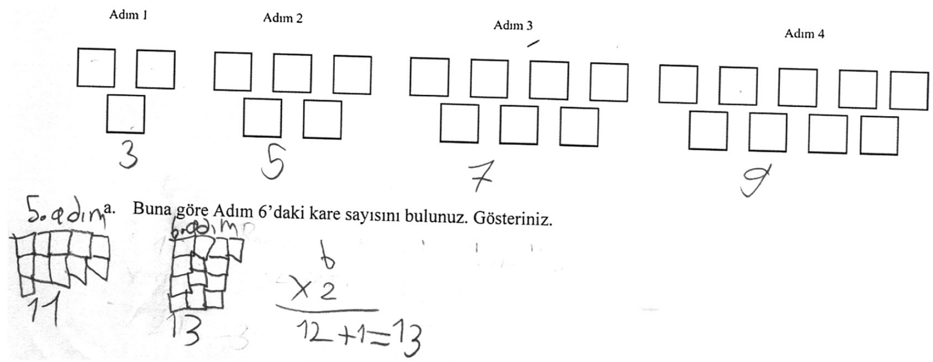
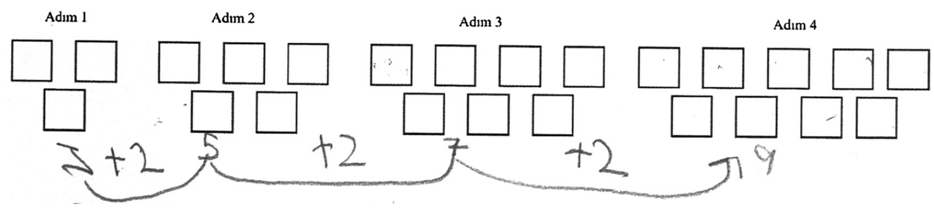
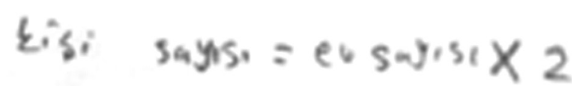
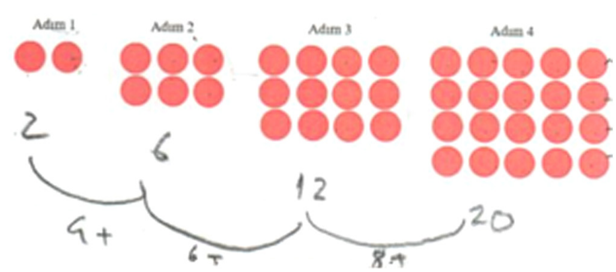


Figure 2.
Data Analysis Scheme of the Study.

Table 3.
Themes, Codes, and Sample Coding

Theme	Code	Sample Coding
Generalization	Modeling	
	Counting	*3 in the 1st step, 5 in the 2nd step, 7 in the 3rd step, 9 in the 4th step, 11 in the 5th step, 13 in the 6th step.*
	Iterative	
	Whole-object	*Since the increment is 2, the rule is "×2."*
	Contextual	
	Linear	
Justification	Explanation	*...So there will be 4 black squares on each side of the 4th step. I drew first, then I counted. The total number of black squares is 16.*
	Algebraic	
	Numerical	*First, I looked at the increase in numbers; +3. So the rule will be "×3," but I need to check whether there is an increase/decrease. First, I tried it in the 1st step; 1 × 3 = 3, but it will be +1 because the result is 4. To find the number of squares, I wrote the rule as "the step where the shape is found ×3 + 1." I also did the operations to find the 50th step.*
	Figural	

Validity and Reliability of the Study

It is required to ensure validity and reliability for qualitative research to have the necessary adequacy (Patton, 2002). In qualitative research, there is credibility in internal validity and transferability in external validity. Internal consistency is sought for reliability, whereas verifiability is sought for external reliability (Lincoln and Guba, 1985). Since the qualitative research method was employed in this study, a long-term study and diversity in data collection tools were used for credibility, purposive sampling and detailed descriptions were used for transferability, and direct quotations were used for verifiability.

Credibility and Transferability of the Study

Validity in qualitative research is addressed with the concepts of credibility and transferability. Credibility is the fact that the data obtained depending on the study purpose are credible. In the present study, teaching experiment activities involving long-term interaction, data diversity obtained from various data sources, and expert review were used to ensure credibility (Lincoln & Guba, 1985).

The researcher was in a long-term interaction with the participants during the study, and thus, the participants' trust in the researcher and their environment was provided. Diversity of data sources is another factor increasing credibility in research. Using this strategy, which is described as diversification, obtaining data with different data collection tools (observation, interview, and document) strengthens the research (Patton, 2002). In the current study, the necessary and adequate data set was obtained by collecting and archiving interviews, audio and video recordings, and written documents.

The transferability of research is its adaptability to similar environments. To adapt this study to similar environments, a detailed and comprehensive description of the research and purposive sampling methods were used. In this study, the transferability of the study results to students with similar characteristics was provided by including participants in the study by purposeful sampling. The content, subject, and scope of the research were explained in detail, and the selection of the participants and the data collection process was presented in detail. Detailed description and direct quotation are essential for transferability. To this end, the transferability of the study was provided by presenting the teaching experiment process and scope, participant characteristics, and the researcher's role in this study (Fraenkel et al., 2006).

Consistency and Verifiability of the Study

In qualitative research, events and situations are dynamic and can be changed and improved. The process of consistently reflecting this variability in research is defined as internal reliability (Lincoln & Guba, 1985). The reliability of the study results, as well as the consistency of the data with each other, is important in terms of consistency, and this consistency is possible with a detailed explanation of each stage of the study (Yin, 2003). Consistency in the present study was provided by explaining the research process in detail and presenting the data collection process in detail.

Verifiability is the objective interpretation of the data obtained as a result of the research by the researcher. Verifiability is provided by a detailed description of the data acquired from various data collection sources. In this study, verifiability was ensured by presenting the details of the coding of the obtained data and the photographs of the student work during the teaching experiment process and supporting the data with direct quotations.

Results

In this section, the results of the teaching experiment applied to improve the generalization and justification strategies of fourth-grade students were analyzed and presented in the context of the generalization and justification strategies in the students' pattern activities. The students' generalization and justification skills that emerged during the teaching experiment carried out to this end are schematized and presented in Figure 3.

As seen in the scheme, the participants carried out activities of creating and continuing patterns in growing patterns and determining the unit of repeat in the teaching experiment conducted in this study. In the process of these activities, the students used multiple representations such as verbal, written, figural/visual, and algebraic. It is observed that the participants generalized in the growing patterns and made justifications in these generalizations, and if parallel to these, differentiating generalization and justification strategies emerged. These thinking structures of the students that emerged in the growing pattern tasks were discussed and evaluated under separate headings.

Results Regarding Pattern Activities

The students carried out activities of creating and continuing patterns and determining the unit of repeat in the growing patterns. As seen in Figure 3, when the frequency of coding is examined, it is observed that the most (number of coding: 80) units of repeat were defined in pattern activities.

In the number patterns, the students were given mixed numbers and asked to create patterns using these numbers. The students' work in the sample pattern and their explanations are given in Table 4.

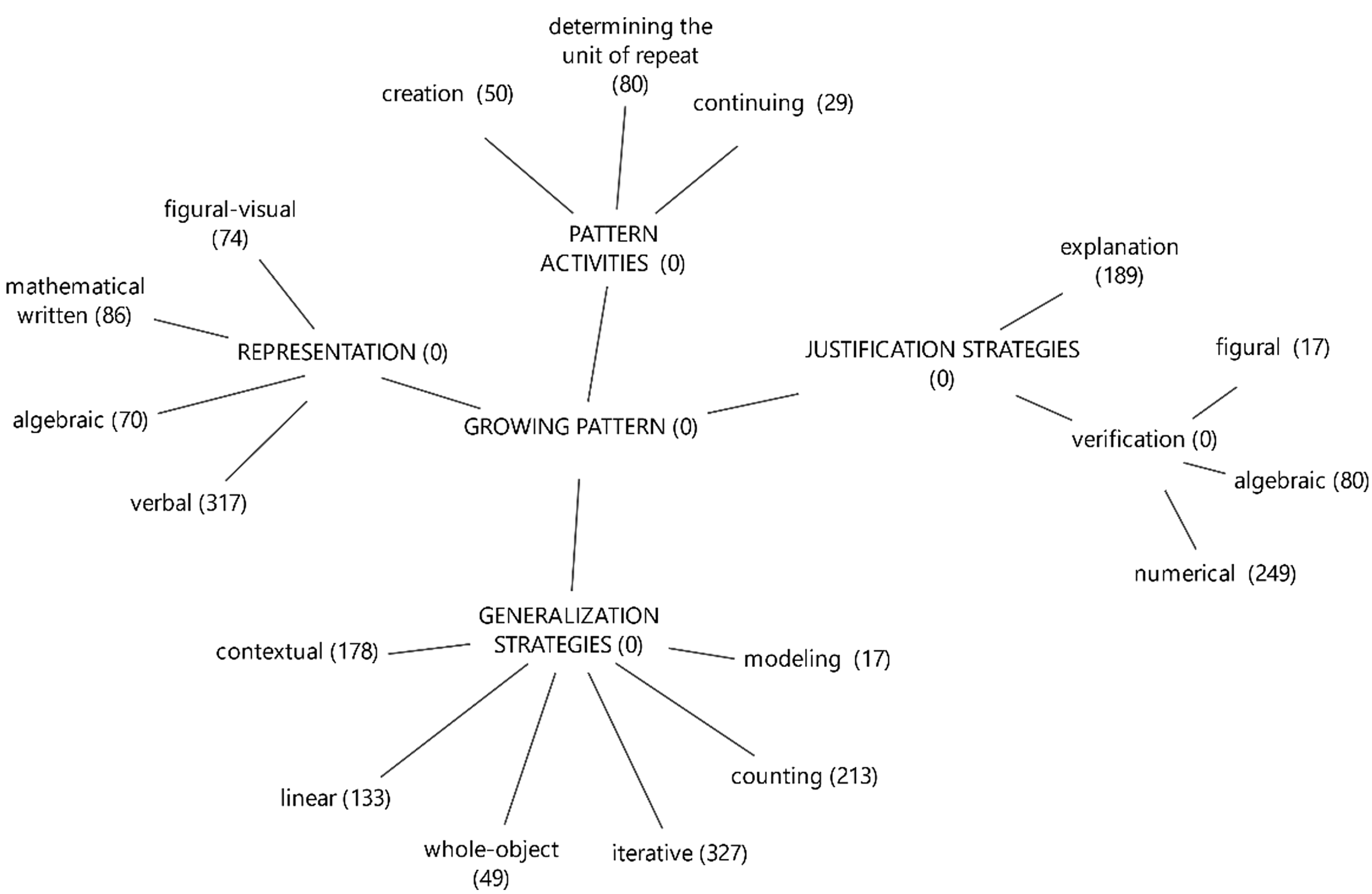
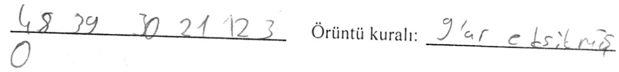
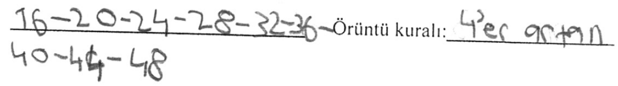
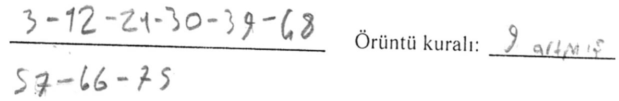
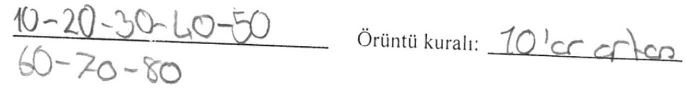


Figure 3.
Students' Generalization and Justification Strategies in the Algebra Teaching Experiment.

Table 4.
Sample Activities of Creating Patterns in Number Patterns

Hale: *I wanted to put the numbers in reverse order. So I chose the numbers 48-39-30-21-12-12-3-0 and put them in order. Thus, the pattern rule here is "descending by 9."*	
Mete: *Differently, I first determined my rule, I said the rule is "increasing by 4." I selected the numbers according to this and put them in order as 12-20-24-….*	
Bade: *Actually, I also first determined my rule and selected the numbers accordingly. If I took random numbers, there would be no pattern. I created the pattern as 3-12-21-30…, so the rule is "increased by 9."*	
Yağız: *I selected the numbers that caught my eye first. The difference between all of them must be the same for these numbers to be a pattern. Therefore, I put the numbers in order as 10-20-30-40-…. If you want, I can continue as 90-100-110.*	

Considering the sample activities carried out in the number patterns in the growing patterns, it is seen that Mete, Bade, and Yağız created the pattern structures based on the condition of being a pattern. These students first determined the pattern rules, then they created the number patterns in line with the rule they determined. It is observed that Hale preferred to create a decreasing pattern, unlike her friends. It is seen that the students could create patterns based on pattern rules and determine the pattern rule in their sample pattern creation activities based on the condition of being/not being a pattern. Yağız also stated that he could continue the pattern he had created. In the growing patterns, the students were asked to continue the patterns given or created by themselves in both number and shape patterns. To this end, the students were asked about the number/s and shape/s that should come in the next steps of the related pattern. The sample pattern continuation activities of the students in the number patterns are given in Table 5.

In the sample pattern activity, the starting point and rule of the pattern were given to the students, and they were asked to create a pattern based on this. In the pattern creation activities of Sarp and Mete, given as an example, the next steps in the sequence were asked, and it was seen that they could continue by applying the rule. It is observed that the students continued the pattern by performing adding or subtracting operations on the number pattern example.

In the growing pattern activities, it is seen that the students define the most (coding number: 80) pattern units of repeat and pattern rules. In the number pattern activities, the students were given a number table of 100, and it was aimed to discover the number patterns and relationships in this table. The sample student explanations regarding this situation are given below.

Mert: *There is a sequence as 24-20-16-12-….*

Researcher: *Is there a rule in the sequence you said?*

Mert: *Yes, decreasing by 4.*

Yağız: *For example, there is 1-11-21-31-vertically… The rule here is increasing by 10. But if we look reversely, we can also write a decrease by 10 from bottom to top as follows: 91-81-71-….*

Hale: *I looked diagonally from left to right, and I saw the pattern 1-12-23-34-45-…. The rule is an increment by 11.*

Bade: *When I look diagonally (from right to left) like Hale did, it continues as 10-19-28-37-…. The rule here is "an increment by 9."*

It was found that the students discovered different number patterns in the presented 100-number table. The students expressed the ordering rule with an increase/decrease in the number sequences they defined. It is observed that the students identified the unit of repeat with an increase/decrease similarly in the shape patterns presented. The sample activities concerning this are presented in Table 6.

When Sarp's solution and explanations are examined, it is seen that he identified the unit of repeat based on the increase/decrease in the numbers of shapes in the steps in the sample pattern. Likewise, in another pattern activity, the students were asked to identify the unit of repeat for the number of pluses and the number of rectangles in the presented visual. In this pattern activity, it is observed that Mete and Hale identified the pattern unit of repeat based on the amount of increase/decrease depending on the order of the number of shapes, as in the number patterns. Asking the questions, *"What did you notice here? What is the iterative situation?"* to the students in each pattern creation and continuation activities revealed that they determined the unit of repeat. The students were supported in identifying the pattern unit of repeat and expressing the rule of the sample situation they were particularly working on and thus reaching generalizations. For this purpose, it was aimed to ensure that the students identified the pattern unit of repeat and/or the pattern rule in their pattern activities. Thus, they were supported in reaching generalizations with regard to the pattern structure and provided with experience.

Table 5.
Sample Pattern Continuation Activities in the Number Patterns

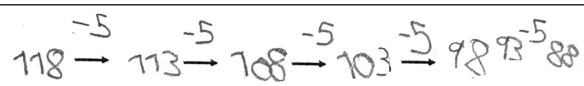
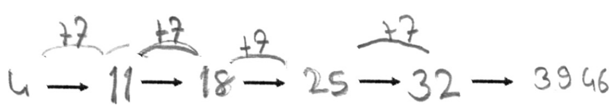
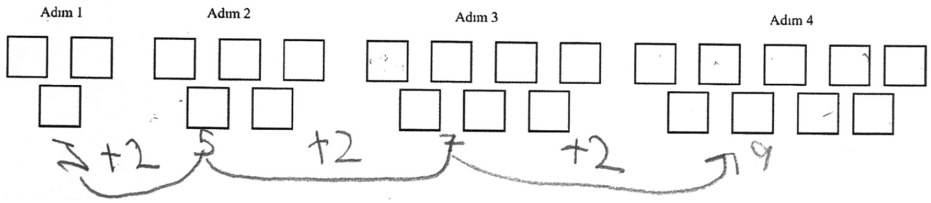
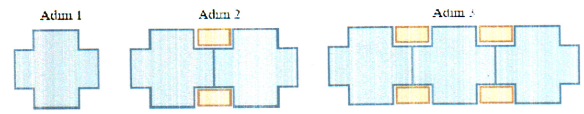
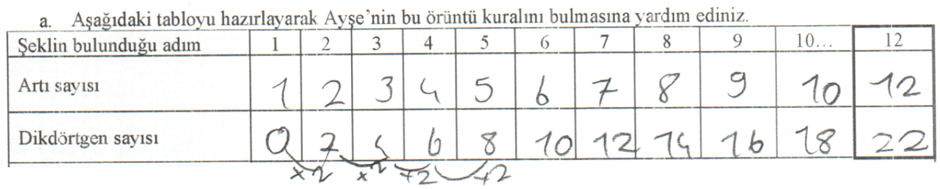
Sarp: *The rule is, "Start from 118 and reduce by 5." That's why, I first started from 118 and subtracted 5; 113. Then I subtracted 5 from 113; 108. So I continued in this way.* Researcher: *Can you continue this pattern for two more steps?* Sarp: *I will continue to subtract; 98-93-88.*	
Mete: *I started with 4 first. Our rule is "increasing by 7." Then it continues as 4-11-18-25-….* *Researcher: So what are the next two steps after 32?* *Mete: 39 and 46.*	

Table 6.
Sample Activities of Identifying the Unit of Repeat in the Shape Patterns

Sarp: *I counted the numbers of the shapes in the step. They are ordered as 3-5-7-9. They increased by 2, from 3 to 5, from 5 to 7, from 7 to 9. Thus, the rule for this is "an increment by 2."*	
Mete: *I first wrote the number of pluses and the number of rectangles in the table. When I looked afterward, I saw this; the number of pluses is 1-2-3-4-5-..., so the rule here is "increasing by 1."*	
Hale: *I also looked at the numbers as Mete did to find the same number of rectangles; the rule is increasing by "2" because it is ordered as 0-2-3-6-8-10-....*	

Results Regarding Generalization and Justification Strategies

It is seen that the students made generalizations in the activities of creating, continuing, and completing patterns and identifying the unit of repeat in the growing pattern tasks. In the scheme concerning the students' generalization and justification strategies (Figure 3), it is observed that they used modeling/counting, iterative/additive, whole-object, contextual, and linear generalization approaches. Additionally, the students made figural, numerical, and algebraic verifications in reaching generalizations and presented their justifications through explanations.

In the sample activity in the students' worksheets, the number of squares in the sixth step, a near step, was asked. The sample solutions and explanations of Sarp and Yağız in this sample shape pattern are given in Figure 4.

> Mete: *I counted the shapes in the step; there are 3 squares in the 1st step, 5 squares in the 2nd step, 7 squares in the 3rd step, and 9 squares in the 4th step. Then there will be 11 squares in the 5th step, and 13 squares in the 6th step. I drew and looked at them.*

> Yağız: *I wrote the number of squares, then I saw there was an increment by 2 between them. If there is an increase by 2, I said, "I will multiply by 2" to find the number in the 6th step. Then I looked whether it was provided in the steps. When I multiply the 1st step by 2, it is 2, but since the number of boxes is 3, I said, "So the number will be plus 1." This also worked when I tried in the other steps. So I did the following, 6 × 2 = 12, 12 + 1 = 13 to find the 6th step.*

In the same pattern task, two different students carried out activities of finding the near step asked in the pattern presented with different generalization and justification strategies. Accordingly, when Mete's solution and explanations are examined, it is seen that he performed modeling and counted the squares in the pattern to find the sixth step. In this way, the student used modeling in generalization and figural control strategies in justification. In his solution, Yağız first identified the unit of repeat (+2) and then expanded this unit of repeat to other steps (×2). After the students performed expansion to the whole, it is observed that he reached a linear function (×2 + 1) by observing the increase/decrease situation. Based on these findings, it is concluded that Yağız used the linear generalization strategy in this sample pattern. In the student's explanations in reaching the linear function, it is seen that he checked the validity of the rule in the other steps and made numerical control by stating that it ensured accuracy. Thus, it is concluded that Yağız made numerical control justification in his generalization regarding the pattern rule in the sample pattern. Likewise, in another pattern activity in which the near step was asked, the sample activities of Bade and Mete were presented in Figure 5, and their generalization and justification strategies were evaluated.

> Bade: *Since the 4th step was asked, I thought I could draw. I looked at the given steps; there is 1 white square in the middle and 1 black square on each side of that white box, 4 black squares in total, in the 1st step. There are 8 black squares in the 2nd step, and 12 black squares in the 3rd step. So there will be 4 black squares on each side of the 4th step. I drew first, then I counted. The total number of black squares is 16.*

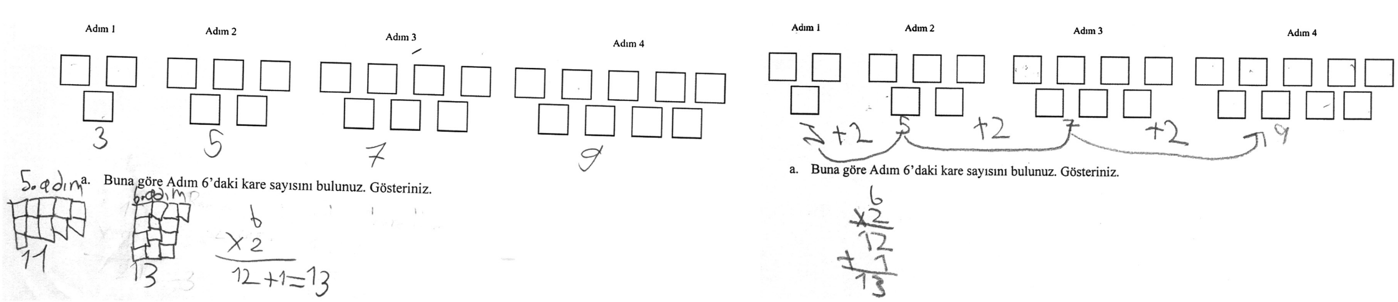


Figure 4.
Sample Activities of the Modeling/Counting and Whole-Object Generalization Strategies.

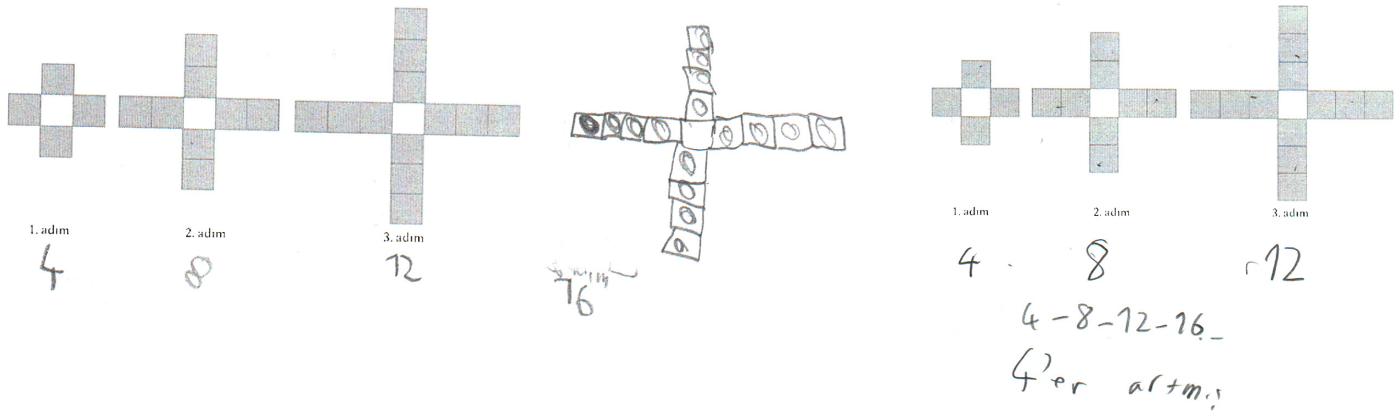


Figure 5.
Sample Activity for the Modeling/Counting Generalization Strategy.

> Mete: *I only looked at the total number of black squares. There are 4 in the 1st step, 8 in the 2nd step, and 12 in the 3rd step. So the sequence will continue as "4-8-12-16-...," i.e., they increased by 4.*

In the sample pattern activity, Bade stated that it would be easier to draw to find the near step, and to this end, she drew by counting the white and black squares in the model. Afterward, she found the number of black squares that should be in the step by counting. In this way, while the student made generalizations based on modeling, she made a figural control in her justification and explained the accuracy of her answer. In addition to this solution, it is observed that Mete found the fourth step based on the amount of increase in the number of black squares in the steps. Concerning this answer of Mete, it is seen that he made justifications through explanations about why it was correct. Another pattern was presented to the students, and whereas Hale could reach a generalization through modeling, Mete generalized with iterative control (Figure 6).

In the sample pattern activity presented, the students were asked the following question, "*The rabbit in the picture can jump from 2 rocks in each jump. Accordingly, from how many rocks will the rabbit have jumped in total in the 8th jump?*" Concerning this question, it is observed that Hale modeled the rock and rabbit jumps in her solution, while Mete made the sequence of the number of rocks jumped. Based on this, it is concluded that Hale reached a generalization by modeling/counting in her pattern activity and Mete reached a generalization with an iterative approach.

In addition to near-step pattern activities, far-step activities were also carried out, and the students' generalization and justification strategies in these patterns were examined. For this purpose, the students' sample activities and explanations of the number and shape patterns are given in Figure 7.

In the presented number pattern activity, the students were supported in performing the table transformation and thus discovering the pattern rule based on the relational situation between the number of steps and the number value. In the pattern in the example, it is seen that Sarp discovered the rule "×5+1" in reaching the number in the pattern starting from the step where the number was found. Sarp's explanations concerning this pattern structure are given below.

> Sarp: *I thought, "How can I reach the number from the step where the number is found." I looked at the increase in the numbers as we did before: 6-11-16-21-26-... The increase is 5, so I said there would be "×5" in the rule. Then I tried this rule in the 1st step; 1 × 5 = 5. To find 6, there will be +1. In other words, the rule must be "×5 + 1." It also worked when I tried this in other steps. I also found the 20th step in this way; 20 × 5 = 100, 100 + 1 = 101.*

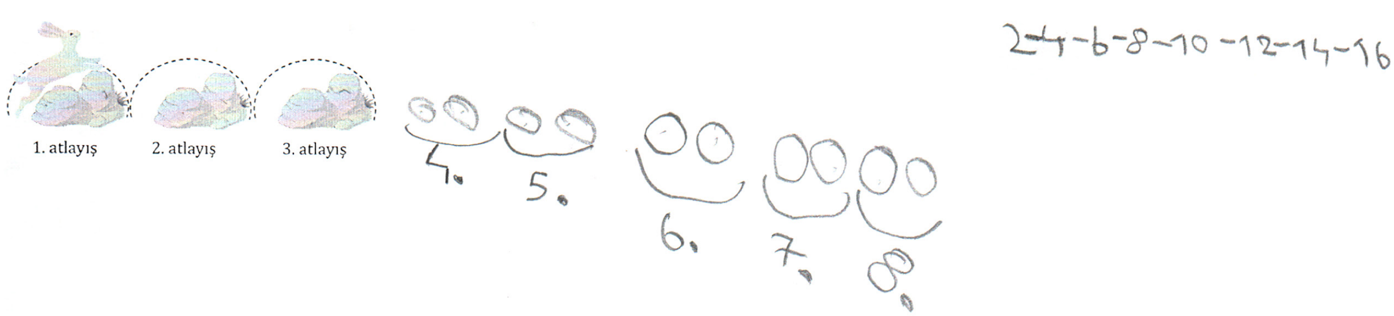
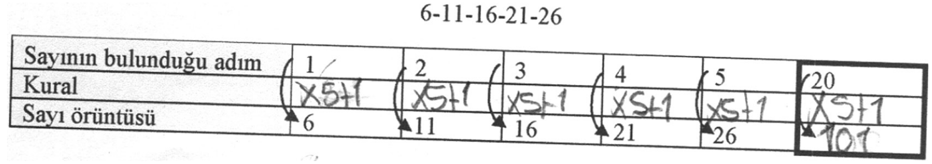


Figure 6.
Sample Activity for the Modeling/Counting and Iterative Generalization Strategies.



6-11-16-21-26

Sayının bulunduğu adım	1	2	3	4	5	20
Kural	X5+1	X5+1	X5+1	X5+1	X5+1	X5+1
Sayı örüntüsü	6	11	16	21	26	101

Figure 7.
Sample Activity for the Linear/Functional Generalization Strategy.

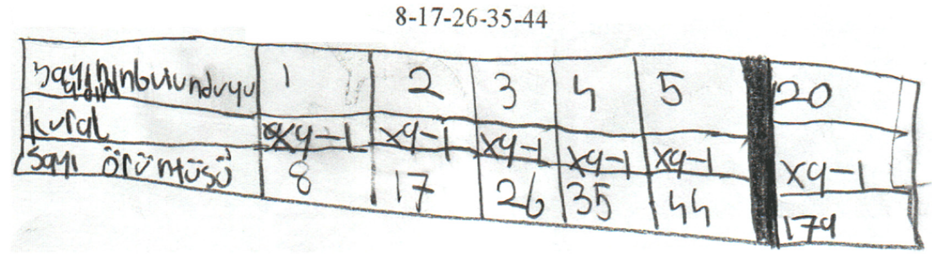


Figure 8.
Sample Activity for the Linear/Functional Generalization Strategy.

When the student's solution and explanations in the sample pattern are examined, it is observed that he discovered the functional rule "×5+1" and found the number in the 20th step based on this rule he discovered. Accordingly, it is concluded that Sarp made justification based on numerical control. In a similar pattern task, the sequence of numbers was given to the students, and the value of the number in the 20th step was asked. The solution and explanations of Yağız concerning this pattern are presented in Figure 8.

> Yağız: *I first wanted to find the rule of this pattern to find the number in the 20th step. I also created a table to find the rule because it's easier to understand. I put the numbers in order by the number of steps as we did before. I first looked at the increase in numbers; 9, then there will be "×9" in the rule. 1 × 9=9, but there is number 8 in the 1st step, so it will be -1. Thus, I said the rule would be "×9 -1" and tried it in other steps. I became sure when I saw this worked. I also applied this rule to find the 20th step, and the result is 129.*

In this sample pattern activity presented, the students were given only the sequence of numbers and asked to find the far step, the 20th step. Considering the activity of Yağız in this pattern, it is observed that he preferred to create a table representation as a facilitator in revealing the relationships between numbers and the relationship between the number of steps and the number value. Using this solution approach, the student reached a generalization of the pattern rule numerically in the form of "×9 - ." It is seen that the student explained the accuracy of this linear function definition by performing numerical operations based on steps. In another shape pattern activity, it is observed that Bade reached generalization in a contextual way. The student's solution and explanations regarding this are presented in Figure 9.

> Bade: *I first looked at the increase in the numbers; +3. So the rule will be "×3," but I need to check whether there is an increase/decrease. I first tried it in the 1st step; 1 × 3=3, but there will be +1 since the result is 4. I wrote the rule as "the step where the shape is found ×3+1" to find the number of squares. I also performed operations to find the 50th step.*

When Bade's solution and explanations in the shape pattern activity presented are examined, it is seen that she first performed expansion to the whole based on the amount of increase and then reached a functional correlation by experimenting with the multiplicative situation with the number of steps - number of squares. It is observed that the student made a contextual generalization without symbolization in the rule of "the step where the shape is found ×3+1" defined by the student. Concerning this generalization, the student revealed its accuracy

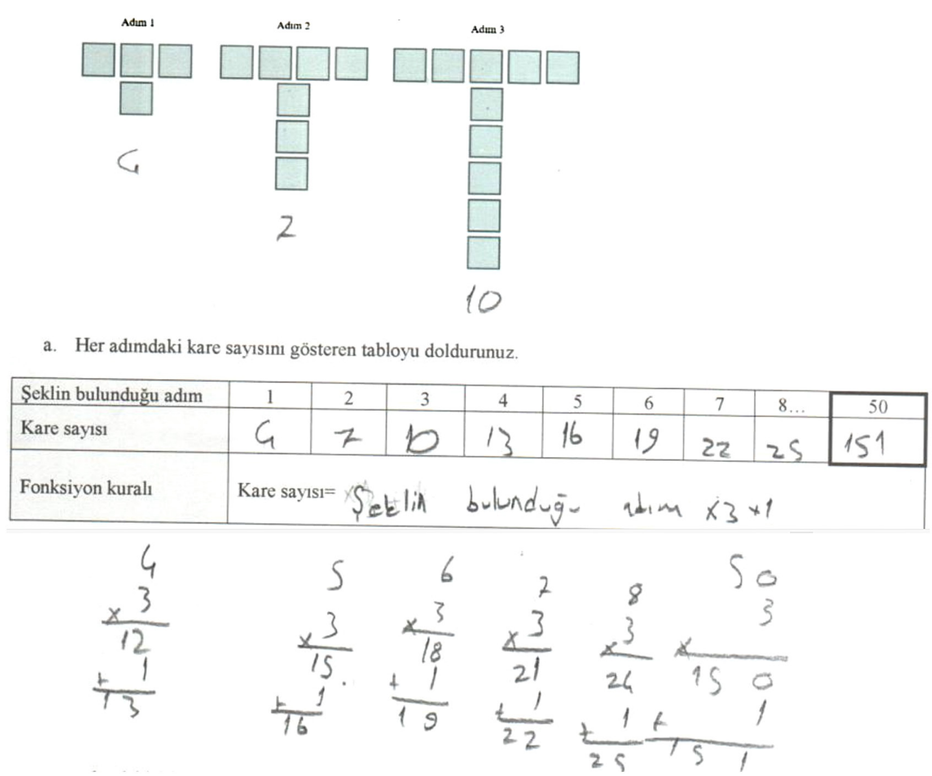


Figure 9.
Sample Activity for the Contextual Generalization Strategy.

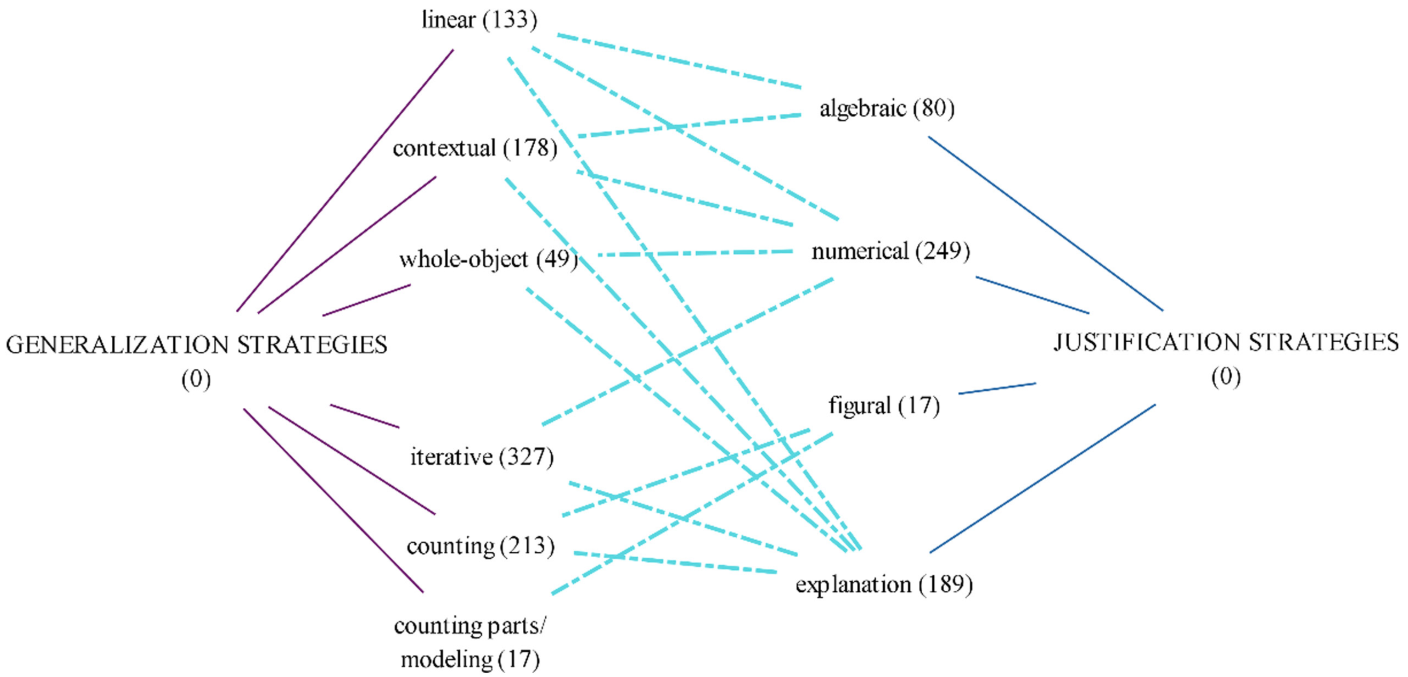


Figure 10.
Relationship Between Generalization and Justification Strategies.

by performing numerical control and justified it through explanations about why it was correct.

Considering the students' generalization and justification strategies in near step and far step pattern activities, it is seen that they used modeling/counting, iterative, and whole-object generalization strategies to find the pattern unit in the near step, whereas they used linear and contextual generalization strategies to find the unit in the far step. Thus, it is concluded that the differentiating pattern activities are effective in revealing the different generalization strategies of students. It is observed that the justification strategies of students differ in parallel with their generalization strategies. It is seen that the students made explanations about figural control or why it was correct in modeling/counting and iterative generalizations. Additionally, they made justifications with numerical or algebraic controls in their contextual or linear generalizations. Based on this, it is concluded that students' generalization strategies and justification ways are related to each other (Figure 10).

Upon examining this schematized structure, it is seen that the linear generalization strategy was justified by the students through algebraic, numerical, control and explanations. When the contextual generalization method, among the generalization strategies, is examined, it is observed to be based on justifications through algebraic, numerical control, and explanations. It is observed that the whole-object generalization strategy was justified by numerical control and explanations, while the students justified with numerical control and explanations in the iterative generalization, justifications were made through figural control and explanations in reaching generalization by counting. Finally, it is seen that only figural justifications were made in the generalization by counting parts/modeling. Based on these results, it is concluded that there is a linear relationship between students' development in generalization strategies and the complexity of their justification strategies.

Discussion, Conclusion, and Recommendations

In the present study, an early algebra teaching experiment was conducted to improve the algebraic thinking structures of fourth-grade students, and the effective generalization and justification strategies of the students during this teaching were revealed. In this section, the effectiveness of the teaching experiment as a result of the students' effective algebraic thinking structures will be discussed by associating it with the literature, and recommendations will be presented.

Growing number and shape patterns were included in the content of the teaching experiment carried out in this study. In growing patterns, the students carried out activities of creating a pattern, completing the missing element, continuing the pattern, creating the pattern with the given rule, and identifying the pattern unit of repeat. It is observed that the students used various representation tools such as written, verbal, and visual in their pattern activities. The use of tables as a supportive teaching element in determining the pattern unit of repeat and reaching the pattern rule was included in the students' number and shape patterns. In parallel to this, the students preferred to perform a table transformation in their solutions, especially in shape patterns, to establish a relationship between the number of shapes depending on the number of steps. Moreover, it is observed that the students developed solutions based on mathematical operations in their solution paths and created linear equations algebraically. Based on these results, it is concluded that the students used multiple representations during the teaching experiment and could perform a transformation between multiple representations. It is also seen that the students' use of multiple representations is a supportive element in discovering the pattern unit of repeat and thus reaching generalizations with regard to the pattern rule. In the teaching experiment applied in the current study, opportunities were provided for the students to reveal mathematical relationships with multiple representations, and it was observed that these mathematical experiences were effective in the students' algebraic thinking structures.

It is seen that the students identified the pattern unit of repeat the most in their pattern activities. In the teaching experiment, questions such as, "What did you notice in the sequence? What are the repeating elements? What is the rule of this sequence?" were asked for the students to identify the unit of repeat in almost all pattern activities. By answering these questions, it was aimed to ensure that the students identified the pattern unit of repeat and thus discovered the pattern rule. The students were supported in identifying the pattern unit of repeat and rule and reaching generalizations in pattern activities.

Algebra and mathematics consist of generalizations (Zazkis & Liljedahl, 2002). Patterns, one of the elements of early algebra, play an active role in ensuring that students at a young age can make generalizations that are important for algebra and mathematics effectively (Tanışlı & Özdaş, 2009). Therefore, justification strategies were examined in the present study in the students' reaching generalizations in number and shape patterns and verifying these generalizations. As a result of these examinations, it was found that the students used counting parts/modeling, counting, iterative, whole-object, linear, and contextual generalization strategies in their pattern activities.

In the pattern activities, the strategies employed by the students in finding the desired term and reaching the generalization with regard to the pattern rule differed in two different situations, the near step and the far step. Whereas the students used modeling/counting, iterative, and whole-object generalization strategies in the pattern activities in which the near step was asked, they preferred to use linear and contextual generalization strategies in finding the far step unit. Carraher et al. (2008) investigated the generalization strategies of third-grade students in patterns containing linear equations. In the study, it was concluded that the majority of young students could find units in the near step in number and shape patterns. Additionally, it was observed in the study that students with advanced algebraic thinking could predict units in far steps and generalize their results both verbally and algebraically. Studies in the literature also demonstrate that students at an early age can make generalizations about mathematical relationships and structures in a way that supports these results (Blanton et al., 2015; Carraher et al., 2008; Martínez & Brizuela, 2006; Stephens et al., 2017; Warren, 2006; Wilkie, 2019). In the study in which Ureña et al. (2019) examined the generalization and representation skills of fourth-grade students, it was revealed that students could notice and represent the generalization of the functional relationship in different ways in situations involving functional relationships. Based on this, it is concluded that asking the near step and far step units in this study is effective in revealing different generalization strategies of students.

Different contexts, such as growing patterns, number, and shape patterns, were included in the teaching experiment. In the number patterns, the students used modeling/counting, iterative, and whole-object strategies based on the amount of increase/decrease according to the order of the numbers. In the shape pattern activities, it was observed that the students tried to reach the result using the numerical values they reached from the shape rather than the structural features of the shapes. This result is supported by the results of similar studies in the literature (Amit & Neria, 2008; Becker & Rivera, 2005). It is seen that the students reached linear and contextual generalizations in their shape pattern activities, unlike number patterns. Accordingly, it is concluded that the pattern activities in the content of the teaching experiment are effective in revealing the different generalization strategies of students.

In the study, when the students' justification strategies in verifying their generalizations are examined, it is observed that they performed figural, numerical, and algebraic control and expressed why the generalizations they reached were correct through explanations. It is seen that figural justifications emerged by drawing and modeling shapes in finding the desired step. Furthermore, the students verified numerically using mathematical operations and expressions. The most advanced justifications were observed when algebraic expressions were used and verifications were made in this way.

In the current study, it is observed that encouraging students to make justifications in pattern activities is effective in the emergence of generalization strategies. Moreover, it was seen that the students' justification ways varied depending on their generalization strategies. In the generalizations reached based on modeling/counting and iterative strategies, figural control or explanations of why it was correct were made, whereas generalizations with numerical or algebraic control were made in contextual or linear generalizations. Accordingly, it is concluded that students' generalization strategies and justification ways are related to each other. Likewise, Ellis (2007) stated that justification affected a student's generalization capability. Thus, it is concluded that students' justification ways influence their generalization strategies.

Generalization, which starts with patterns in primary and secondary school, is the heart of algebraic thinking (Blanton & Kaput, 2004; Carraher et al., 2008; Warren & Cooper, 2008). Hence, in this study, growing number and shape patterns including near step and far step generalizations were addressed, and the effectiveness of these patterns in students' generalization and justification strategies was investigated. Furthermore, the relational situation in the students' generalization and justification strategies was discussed. The findings demonstrate that the teaching experiment carried out in this study allowed students to reach generalizations with different strategies and justify these generalizations in different verification ways. It is found that generalizations and justifications are effective since students are introduced with the number and shape patterns in the teaching experiment and acquire experience with various contexts.

The findings obtained from the present study support the available information in the literature. Blanton et al. (2015) indicated that early algebraic thinking practices such as generalizing, representing, and reasoning mathematical relationships and structures could be integrated into early grade levels. Likewise, as a result of the teaching experiment they conducted with students in the pre-algebra period, Stephens et al. (2017) demonstrated that students' ability to determine the general rules between numbers improved and the rules of patterns involving relational situations could be expressed by students with symbols and words. In the teaching experiment conducted in this study, it was concluded that students in the early algebra period could generalize and justify in growing number and shape patterns. Recommendations were made for future studies in line with the findings and results obtained from the research.

- The effectiveness of the teaching experiment applied in the study is limited within the scope of the individual characteristics of the students participating in the teaching experiment. Therefore, the effectiveness of the teaching content can be evaluated with the same teaching experiment to be carried out with students having different characteristics. Primary school fourth-grade students in the early algebra period took part in the study. The effectiveness of students' algebraic thinking structures can be evaluated by applying the teaching experiment with students in the early algebra period in different classes and of different ages.
- Linear patterns were included in the teaching experiment carried out in this study. The effectiveness of relational situations in this form of patterns on students' generalization and justification strategies was evaluated. In future studies, linear patterns can be differentiated, and their effectiveness can be revealed, and thus a pool of growing number and shape patterns effective in early algebra teaching can be created.
- The effectiveness of students' algebraic thinking structures can be evaluated by integrating technology tools in early algebra teaching practices planned to be carried out in the future.
- In the present study, the effectiveness of the early algebra teaching experiment was evaluated within the scope of students' generalization and justification strategies. In the studies to be performed, the situations of making sense of the concepts of symbolization, variable, equality, and equivalence can be examined in the context of students' algebraic thinking.

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References

Amit, M., & Neria, D. (2008). "Rising to the challenge": Using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students. *ZDM*, *40*(1), 111–129. [CrossRef]

Ayala-Altamirano, C., & Molina, M. (2021). Fourth-graders' justifications in early algebra tasks involving a functional relationship. *Educational Studies in Mathematics*, *107*(2), 359–382. [CrossRef]

Balacheff, N. (2001). Symbolic arithmetic vs algebra: The core of a didactical dilemma. In R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds.), *Perspectives on school algebra* (pp. 249–260). Kluwer Academic.

Becker, J. R., & Rivera, F. (2005). Generalization strategies of beginning high school algebra students. In H. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the International Group for the Psychology of Mathematics Education* (pp. 121–128). University of Melbourne.

Blanton, M. (2008). *Algebra and the elementary classroom: Transforming thinking, transforming practice*. Pearson Education.

Blanton, M. L., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in six-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*, *46*(5), 511–558. [CrossRef]

Blanton, M. L., & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 5–23). Springer.

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2017). A progression in first-grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics*, *95*(2), 181–202. [CrossRef]

Blanton, M., & Kaput, J. (2004). Elementary grades students' capacity for functional thinking. In M. Jonsen Hoines & A. Hoines (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (pp. 135–142). Bergen University College.

Blanton, M., Levi, L., Crites, T., & Dougherty, B. (2011). *Developing essential understanding of algebraic thinking for teaching mathematics in grades 3–5*. National Council of Teachers of Mathematics.

Blanton, M., Stroud, R., Stephens, A., Gardiner, A. M., Stylianou, D. A., Knuth, E., Isler-Baykal, I., & Strachota, S. (2019). Does early algebra matter? The effectiveness of an early algebra intervention in grades 3 to 5. *American Educational Research Journal*, *56*(5), 1930–1972. [CrossRef]

Boyer, C. B. (1991). *A history of mathematics* (2nd ed). Wiley.

Breiteig, T., & Grevholm, B. (2006). The transition from arithmetic to algebra: To reason, explain, argue, generalize and justify. *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (pp. 225–232). PME.

Cai, J., & Knuth, E. (Eds.) (2011). *Early algebraization: A global dialogue from multiple perspectives*. Springer.

Cañadas, M. C., Brizuela, B. M., & Blanton, M. (2016). Second graders articulating ideas about linear functional relationships. *Journal of Mathematical Behavior*, *41*, 87–103. [CrossRef]

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Heinemann.

Carraher, D. W., & Schliemann (2007). Early algebra and algebraic reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669–705). National Council of Teachers of Mathematics.

Carraher, D. W., & Schliemann, A. D. (2015). Powerful ideas in elementary school mathematics. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 199–218). Taylor & Francis.

Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). Lawrence Erlbaum Associates.

Chimoni, M., Pitta-Pantazi, D., & Christou, C. (2018). Examining early algebraic thinking: Insights from empirical data. *Educational Studies in Mathematics*, *98*(1), 57–76. [CrossRef]

Cobb, P. (2000). Conducting teaching experiment in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–333). Lawrence Erlbaum Publishers.

Cobb, P., & Steffe, L. (2011). *A journey in mathematics education research*. Springer.

Cooper, T. J., & Warren, E. (2011). Years 2 to 6 students' ability to generalise: Models, representations and theory for teaching and learning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 187–214). Springer.

Dienes, Z. P. (1961). On abstraction and generalization. *Harvard Educational Review*, *31*, 281–301.

Doğan Temur, Ö., & Turgut, S. (2020). Sınıf Öğretmenleri Sayı ve Şekil Örüntülerini Nasıl Öğretiyorlar? İlkokul Dördüncü Sınıf Örneği. *Abant İzzet Baysal Üniversitesi Eğitim Fakültesi Dergisi*, *20*(1), 182–200. [CrossRef]

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, *25*(4), 439–478. [CrossRef]

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 215–238). Springer.

Engelhardt, P. V., Corpuz, E. G., Ozimek, D. J., & Rebello, N. S. (2004). The teaching experiment- what it is and what it isn't. In Physics Education Research Conference, 2003 (pp. 157–160).

Fonger, N. L., Stephens, A., Blanton, M., Isler, I., Knuth, E., & Gardiner, A. M. (2018). Developing a learning progression for curriculum, instruction, and student learning: An example from mathematics education. *Cognition and Instruction*, *36*(1), 30–55. [CrossRef]

Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2006). *How to design and evaluate research in education*. McGraw Hill.

Gattegno, C. (1973). In the beginning there were no words: The universe of babies. *Educational Solutions*.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DRN based instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and teaching number theory, journal of mathematical behavior* (pp. 185–212). Albex.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), *Research in collegiate mathematics education* (pp. 234–283). American Mathematical Society.

Herbert, K., & Brown, R. H. (1997). Patterns as tools for algebraic reasoning. *Teaching Children Mathematics*, *3*(6), 340–344. [CrossRef]

Jurow, A. S. (2004). Generalizing in interaction: Middle school mathematics students making mathematical generalizations in a population-modeling project. *Mind, Culture, and Activity*, *11*(4), 279–300. [CrossRef]

Kabael, T. U., & Tanışlı, D. (2010). Teaching from patterns to functions in algebraic thinking process. *Elementary Education Online*, *9*(1), 213–228.

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Lawrence Erlbaum Associates.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). Routledge.

Kaput, J. J., & Blanton, M. (2001). Algebrafying the elementary mathematics experience. Part 1: Transforming task structures. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: the Future of the teaching and learning of algebra* (pp. 344–351). The University of Melbourne.

Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19–55). Lawrence Erlbaum Associates.

Kieran, C. (2011). Overall commentary on early algebraization. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 579–594). Springer.

Kirwan, J. V. (2015). *Preservice secondary mathematics teachers' knowledge of generalization and justification on geometric numerical patterning tasks* [Doctoral Dissertation]. Illinois State University.

Knuth, E., Slaughter, M., Choppin, J., & Sutherland, J. (2002). Mapping the conceptual terrain of middle school students' competencies in justifying and proving. In D. Mewborn, et al. (Eds.), *Proceedings of the twenty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1693–1700). PME-NA.

LaCampagne, C. B., Blair, W., & Kaput, J. (1995). *The algebra initiative colloquium*. Department of Education.

Lannin, J. K. (2005). Generalization and justification the challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning*, *7*(3), 231–258. [CrossRef]

Linchevski, L. (1995). Algebra with numbers and arithmetic with letters: A definition of pre-algebra. *Journal of Mathematical Behavior*, *14*(1), 113–120. [CrossRef]

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Sage Publishing.

Martínez, M., & Brizuela, B. M. (2006). A third grader's way of thinking about linear function tables. *Journal of Mathematical Behavior*, *25*(4), 285–298. [CrossRef]

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and practice* (pp. 65–86). Springer.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. Kaput, D. Carraher & M. Blanton (Eds.), *Algebra in the early grades* (pp. 57–94). Lawrence Erlbaum Associates & National Council of Teachers of Mathematics.

Mason, J. (2018). How early is too early for thinking algebraically? In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds* (pp. 329–350). Springer.

Mason, J., Johnston-Wilder, S., & Graham, A. (2005). *Developing thinking in Algebra*. Sage.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*(3), 277–289. [CrossRef]

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematics structure for all. *Mathematics Education Research Journal*, *21*(2), 10–32. [CrossRef]

Merriam, S. B. (1998). *Qualitative research and case studies applications in education*. Jossey-Bass Publications.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. National Council for Teachers of Mathematics.

Palabıyık, U., & İspir, O. A. (2011). The effects of pattern-based algebra instruction on students' algebraic thinking and attitude towards mathematics. *Pamukkale University Journal of Education*, *30*(30), 111–123.

Patton, M. Q. (2002). *Qualitative research & evaluation methods*. Sage Publications.

Pinto, E., & Cañadas, M. C. (2021). Generalizations of third and fifth graders within a functional approach to early algebra. *Mathematics Education Research Journal*, *33*(1), 113–134. [CrossRef]

Pinto, E., Cañadas, M. C., & Moreno, A. (2022). Functional relationships evidenced and representations used by third graders within a functional approach to early algebra. *International Journal of Science and Mathematics Education*, *20*(6), 1183–1202. [CrossRef]

Radford, L. (2003). Gestures, Speech, and the Sprouting of Signs: A Semiotic-Cultural Approach to Students' Types of Generalization [Speech]. *Mathematical Thinking and Learning*, *5*(1), 37–70. [CrossRef]

Radford, L. (2006). Algebraic thinking and the generalization of pattern: A semiotic perspective. In *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 2–21). Universidad Pedagógica National.

Ramírez, R., Brizuela, B. M., & Ayala-Altamirano, C. (2020). Word problems associated with the use of functional strategies among grade 4 students. *Mathematics Education Research Journal*, 1–25. [CrossRef]

Rivera, F. (2013). *Teaching and learning patterns in school mathematics: Psychological and pedagogical considerations*. Springer.

Schifter, D. (2018). Early algebra as analysis of structure. A focus on structure. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds* (pp. 309–327). Springer.

Schifter, D., Monk, S., Russell, S. J., & Bastable, V. (2008). What does understanding the laws of arithmetic mean in the elementary grades? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 413–448). Routledge.

Sinclair, H. (1987). Constructivism and the psychology of mathematics. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th PME international conference* (pp. 28–41).

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, *20*(2), 147–164. [CrossRef]

Steffe, L. P. (1983). The teaching experiment methodology in a constructivist research program. In M. Zweng, T. Green, J. Kilpatrick, H. Pollack, & M. Suydam (Eds.), *Proceedings of the fourth international congress on mathematical education* (pp. 469–471). Birkhauser Verlag.

Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning*, *19*(3), 143–166. [CrossRef]

Subramaniam, K., & Banerjee, R. (2011). The arithmetic-algebra connection: A historical-pedagogical perspective. In J. Cai & D. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 87–107). Springer.

Tanışlı, D., & Özdaş, A. (2009). The Strategies of Using the Generalizing Patterns of the Primary School 5th Grade Students. *Educational Sciences: Theory and Practice*, *9*(3), 1485–1497.

Threlfall, J. (1999). Repeating patterns in the early primary years. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 18–30). Cassell.

Turgut, S., & Doğan Temur, Ö. (2020). Determining elementary teachers ıdeas about early algebra 2. *Elementary Education Online*, *16*(4), 1469–1490. [CrossRef]

Türkmen, H., & Tanışlı, D. (2019). Cebir Öncesi: 3, 4 ve 5. Sınıf Öğrencilerinin Fonksiyonel Ilişkileri Genelleme Düzeyleri. *Eğitimde Nitel Araştırmalar Dergisi*, *7*(1), 344–372.

Ureña, J., Ramírez, R., & Molina, M. (2019). Representations of the generalization of a functional relationship and the relation with the interviewer's media tion/Representaciones de la generalización de una relación funcional y el vínculo con la mediación del entrevistador. *Infancia y Aprendizaje*, *42*(3), 570–614. [CrossRef]

Vergel, R. (2015). Generalización de Patrones y Formas de Pensamiento Algebraico Temprano [Pattern generalization and forms of early algebraic thinking]. *PNA Special Issue on Generalization*, *9*(3), 193–215.

Warren, E. (2006). Comparative mathematical language in the elementary school: A longitudinal study. *Educational Studies in Mathematics*, *62*(2), 169–189. [CrossRef]

Warren, E., & Cooper, T. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics*, *67*(2), 171–185. [CrossRef]

Wilkie, K. J. (2019). Investigating secondary students' generalization, graphing, and construction of figural patterns for making sense of quadratic functions. *Journal of Mathematical Behavior*, *54*, 100689. [CrossRef]

Xolocotzin, U., & Rojano, T. (2015). The development and arithmetic foundations of early functional thinking. In *9th Congress of European Research in Mathematics Education*. ERME.

Yin, R. K. (2003). *Applications of case study research*. Sage Publishing.

Zazkis, R., & Liljedahk, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, *49*(3), 379–402. [CrossRef]

Appendix 1

Sample Focus Group Interview Questions

1. Examine the numbers in the sequence. How are the numbers/shapes ordered?
2. By how many numbers do the numbers continue to increase/decrease according to the number relationship you specified?
3. Is the increase/decrease between each number the same?
4. What are the numbers not given in the pattern?
5. Can you explain how you found the number that was not given and continued the pattern?
6. How would you express the pattern rule?
7. What is the relationship between the numbers in the pattern?
8. What did you pay attention to when determining the relationships between the numbers?
9. How would you express the pattern rules? Can you find another pattern rule as an alternative to the pattern rule you specified?
10. Please explain how you obtained this rule.